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INCLUSIONS AND INHOMOGENEITIES IN TRANSVERSELY ISOTROPIC PIEZOELECTRIC SOLIDS

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Abstract—We analyse the electroelastic fields in and around inclusions and inhomogeneities in transversely isotropic piezoelectric solids using Eshelby's pioneering approach. Following a brief review of the general theory, we obtain explicit, closed-form expressions for the four tensors that are the piezoelectric analog of Eshelby's tensor for *spheroidal* inclusions in a transversely isotropic piezoelectric medium. We focus on no specific problem pertaining to piezoelectric inclusions and inhomogeneities, but instead provide an easily-used general solution. The explicit expressions for the four tensors can be used, in the same manner as Eshelby's tensor for elastic inclusions, to solve a wide range of problems in the mechanics and physics of heterogeneous piezoelectrics. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Eshelby's (1957, 1959) classical analyses of the stress and strain fields in elastic solids containing ellipsoidal inclusions and inhomogeneities are widely recognized both for their elegance and wide-ranging applicability. Indeed, Eshelby's method and his results serve as the cornerstone of many contemporary micromechanics studies of defects, fracture, and the behavior of heterogeneous media at various length scales. Numerous examples of and references to such applications can be found in the texts of Mura (1987) and Nemat-Nasser and Hori (1993). Eshelby provided many useful results including: demonstration of the uniformity of stress and strain fields in ellipsoidal inclusions with uniform eigenstrains and ellipsoidal inhomogeneities subjected to uniform far-field loads, the equivalent inclusion method, and simple, efficient methods for energy calculations. Perhaps the most widely-used result of Eshelby's analyses is his simple, closed-form expression for what is now known as Eshelby's tensor: a fourth-order tensor which is a function only of the elastic moduli of the matrix and the shape of the inclusion. In fact, with the explicit expressions for Eshelby's tensor in hand, solutions to many problems concerning inclusions and inhomogeneities are reduced to algebraic tensor manipulation. While Eshelby only provided explicit results for inclusions in isotropic solids, he laid the groundwork for the study of inclusions in anisotropic solids. Subsequent researchers (Hill (1971), Willis (1964), Walpole (1967, 1977), Kinoshita and Mura (1971), Lin and Mura (1973), Asaro and Barnett (1975), Bacon *et al.*, (1978), among others) provided valuable results regarding inclusions in anisotropic matrices. The key component, Eshelby's fourth order tensor, was expressed in terms of surface integrals over a unit sphere or line integrals along a unit circle. As no closed form expressions were obtained, Eshelby's tensor had to be computed by numerical integration (see, for example, Gavazzi and Lagoudas, 1990). Only for transversely isotropic solids are analytical results for Eshelby's tensor available (Withers, 1989; Yu *et al.*, 1994). It is not surprising that the use of Eshelby's tensor for anisotropic solids pales in comparison to its use for isotropic solids. There are probably two reasons for this: isotropic materials play a more prominent role in technological applications than anisotropic ones, and anisotropic analysis is often considered to be somewhat more complex than isotropic analysis.

When dealing with piezoelectric solids, transverse isotropy is of fundamental importance: the most technologically-important piezoelectric materials are poled ceramics which exhibit transverse isotropy with the unique axis aligned along the poling direction. Piezoelectric inclusions and inhomogeneities have been studied by numerous researchers (Deeg, 1980; Wang, 1992; Benveniste, 1992; Dunn and Taya, 1993; Chen, 1993a, b). Deeg, Dunn and Taya used a direct generalization of Eshelby's elegant approach, while Benveniste and Chen generalized the approaches of Walpole (1967) and Hill (1961). Nevertheless, they all obtained expressions (although none in closed form) for the four tensors that comprise the piezoelectric analog of Eshelby's tensor in elasticity. Dunn and Taya obtained expressions for these tensors in terms of surface integrals over the unit sphere which they evaluated numerically, and Dunn (1994) obtained closed-form expressions for the tensors in the case of elliptical cylindrical inclusions in transversely isotropic solids. To date, however, no closed-form expressions have been obtained for the piezoelectric Eshelby tensors for *spheroidal* inclusions (which can simulate inclusion geometries ranging from thin disks to long needles) in transversely isotropic solids. The development of such expressions is the objective of this study.

To this end, the basic equations of linear piezoelectricity, a convenient shorthand notation, and a brief review of the solution of inclusion and inhomogeneity problems in piezoelectric solids are given in Section 2. The main ingredients of the present solution, the piezoelectric Green's functions, are presented in Section 3. In Section 4 we define and then derive explicit closed-form expressions for the piezoelectric Eshelby tensors for spheroidal inclusions in transversely isotropic media. These expressions can be trivially simplified for the cases of disk-shaped, spherical, and needle-shaped inclusions. Our approach proceeds in a manner that parallels Eshelby's derivation. This is a departure from most analyses involving anisotropic media, and all approaches involving piezoelectric media, which make use of transform formalism and yield results in terms of surface integrals over the unit sphere. We emphasize that the intent of this work is not to study any one particular aspect of piezoelectric inclusions and inhomogeneities in detail. Rather, we explicitly provide the general solution; specifically, the piezoelectric Eshelby tensors which can be used with the standard Eshelby approach. Our results can be easily and immediately used by researchers interested in pursuing specific applications.

2. INCLUSIONS AND INHOMOGENEITIES IN LINEAR PIEZOELECTRICITY

In this section we review the basic equations of linear piezoelectricity and the analysis of inclusion and inhomogeneity problems in piezoelectric solids; our explicit expressions for the piezoelectric Eshelby tensors can be easily used with these. Most of the equations presented in this section have appeared in the literature, thus we omit derivation and provide appropriate references. We consider an ellipsoidal inclusion or inhomogeneity and focus on the uniform (Deeg, 1980) interior electroelastic fields. We do not treat the complicated electroelastic fields outside the inclusion or inhomogeneity (except just at the boundary). It is the interior fields that are most important, as with them alone we can tackle many problems in heterogeneous media.

Basic equations

A three-dimensional cartesian coordinate system is employed where position is denoted by the vector $\mathbf{x}$ or x_i. In this paper, both indicial x_i and cartesian x, y, z notations are utilized. For stationary behavior in the absence of free electric charge or body forces, the field equations of linear piezoelectricity consist of the constitutive equations, the divergence equations (elastic equilibrium and Gauss' law), and the gradient equations (strain–displacement and electric field–potential relations). In full index form these are:

$$\begin{aligned} \sigma_{ij} &= C_{ijmn}\varepsilon_{mn} - e_{nij}E_n \\ D_i &= e_{imn}\varepsilon_{mn} + \kappa_{in}E_n \end{aligned} \quad (1)$$

$$\begin{aligned} \sigma_{ij,j} &= 0 \\ D_{i,i} &= 0 \end{aligned} \quad (2)$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$$

$$E_i = -\phi_{,i}. \tag{3}$$

In eqns (1)–(3), σ_{ij}, ε_{ij} and u_i are the elastic stress, strain, and displacement, respectively; D_i, E_i and ϕ are the electric displacement, field, and potential, respectively; C_{ijmn}, e_{nij}, and κ_{in} are the elastic stiffness tensor (measured in a constant electric field), the piezoelectric tensor, and the dielectric tensor (measured at a constant strain), respectively. The symmetry conditions satisfied by the electroelastic moduli are given by Nye (1957), and C_{ijmn} and κ_{in} are positive definite.

In linear piezoelectric analysis, it is convenient to treat the elastic and electric variables on equal footing. To this end, the notation introduced by Barnett and Lothe (1975) is utilized. This notation is identical to conventional indicial notation with the exception that lowercase subscripts take on the range 1, 2, 3, while uppercase subscripts take on the range 1, 2, 3, 4. With this notation, the field variables take the following forms:

$$U_M = \begin{cases} u_m \\ \phi \end{cases} \quad Z_{Mn} = \begin{cases} \varepsilon_{mn} \\ \phi_{,n} \end{cases} \quad \Sigma_{nM} = \begin{cases} \sigma_{nm} & M = 1,2,3 \\ D_n & M = 4 \end{cases}. \tag{4}$$

The electroelastic moduli are expressed as:

$$E_{iJMn} = \begin{cases} C_{ijmn} & J,M = 1,2,3 \\ e_{nij} & J = 1,2,3; \quad M = 4 \\ e_{imn} & J = 4; \quad M = 1,2,3 \\ -\kappa_{in} & J,M = 4 \end{cases}. \tag{5}$$

With this shorthand notation, the constitutive equations can be written as $\Sigma_{iJ} = E_{iJMn}Z_{Mn}$. Ten material constants are required to describe a transversely isotropic piezoelectric solid with the x_3 (z) axis normal to the plane of isotropy: five elastic ($C_{11}, C_{13}, C_{33}, C_{44}, C_{66}$), three piezoelectric ($e_{31}, e_{33}, e_{15}$), and two dielectric ($\kappa_{11}, \kappa_{33}$). Here we have employed the well-known Voigt two-index notation.

Piezoelectric inclusions and inhomogeneities

Consider an infinite piezoelectric solid D containing an ellipsoidal inclusion denoted by Ω with surface $|\Omega|$. The inclusion has the same electroelastic moduli, E_{iJMn}, as the matrix, but undergoes a uniform electroelastic transformation (which may, for example, be associated with the spontaneous polarization and deformation that occur during a crystallographic phase transformation). We denote by Z^*_{Mn} the uniform transformation that would occur if Ω were unconstrained by D. To calculate the actual (constrained) electroelastic fields, the imaginary cutting, straining, and welding operations of Eshelby (1957) can be utilized. As has been shown by Deeg (1980), Benveniste (1992), and Dunn and Taya (1993), we can express the uniform (because of the ellipsoidal shape) stress and electric displacement in the inclusion as:

$$\Sigma_{iJ} = E_{iJMn}[Z_{Mn}-Z^*_{Mn}]. \tag{6}$$

Due to linearity, the strain and electric field in the inclusion can be expressed in terms of Z^*_{Mn} by the introduction of the piezoelectric Eshelby tensors S_{MnAb}:

$$Z_{Mn} = S_{MnAb}Z^*_{Ab}. \tag{7}$$

Formally, S_{MnAb} is a collection of four tensors: one fourth-order, one second-order, and two third-order. S_{MnAb} is a function only of the electroelastic moduli, the shape of the inclusion, and the orientation of the inclusion relative to the principal material axes. If the Eshelby tensors S_{MnAb} are known, then for prescribed eigenfields Z^*_{Mn}, the constrained electroelastic fields in the inclusion can immediately be computed with eqns (6) and (7). Equation (7) and the forthcoming results are based on the idea of a transformation strain and potential gradient, i.e. Z^*_{Mn}. In many cases, it is more convenient to deal with the transformation stress and electric displacement Σ^*_{iJ} or a combination of Z^*_{Mn} and Σ^*_{iJ}. An example is in the analysis of crystallographic phase transformations in piezoelectric solids. The unconstrained phase transformation is accompanied by a spontaneous strain and polarization. These can be directly represented by transformation quantities: the former by ε^*_{mn} and the latter by D^*_i, and their incorporation in the analysis is straightforward.

Once the solution for the ellipsoidal inclusion (a transformed region with the same electroelastic moduli as the matrix) is obtained, the solution for the ellipsoidal inhomogeneity (a region with different electroelastic moduli than the matrix) easily follows. As shown by Eshelby (1957) in the elastic case and Deeg (1980) in the piezoelastic case, the inhomogeneity can be simulated by an *equivalent inclusion*. To fix ideas, consider the infinite piezoelectric solid D with electroelastic moduli E_{iJMn} which contains an ellipsoidal inhomogeneity Ω with electroelastic moduli E^*_{iJMn}. In the absence of an applied electrical or mechanical load, the electroelastic fields in both the inhomogeneity and matrix are zero. When subjected to a far-field uniform load Σ^0_{iJ}, the stress and electric displacement in the inhomogeneity, $\Sigma^0_{iJ}+\Sigma_{iJ}$, can be written as:

$$\Sigma^0_{iJ}+\Sigma_{iJ} = E^*_{iJMn}[Z^0_{Mn}+Z_{Mn}] = E_{iJMn}[Z^0_{Mn}+Z_{Mn}-Z^*_{Mn}]. \tag{8}$$

In eqn (8), Z^0_{Mn} is the uniform strain and potential gradient that would exist in the absence of the inhomogeneity ($\Sigma^0_{iJ} = E_{iJMn}Z^0_{Mn}$) and Z_{Mn} is the disturbance of the uniform fields due to the presence of the inhomogeneity. The first right-hand side of eqn (8) represents the stress and electric displacement in the actual inhomogeneity while the second one represents the stress and electric displacement in an inclusion of the same shape and orientation as the inhomogeneity and with eigenfields Z^*_{Mn}, i.e. an *equivalent inclusion*. The simulation of the inhomogeneity by the equivalent inclusion is possible if an appropriate Z^*_{Mn} can be found to enforce the second equality of eqn (8) (where eqn (7) holds in the equivalent inclusion). Substituting eqn (7) into eqn (8), and solving for Z^*_{Mn} gives

$$Z^*_{Pq} = -A^{-1}_{PqiJ}[E^*_{iJMn}-E_{iJMn}]Z^0_{Mn} \tag{9}$$

where $A_{iJAb} = [E^*_{iJMn}-E_{iJMn}]S_{MnAb}+E_{iJAb}$. Once $Z^*_{Pq}(Z^0_{Mn})$ is obtained from (9), it can be used with eqns (7) and (8) to obtain the electroelastic fields in the inhomogeneity due to the applied electroelastic load. Thus the inclusion is equivalent in the sense that (for an eigenfield history $Z^*_{Pq}(Z^0_{Mn})$) its electroelastic field history mirrors that of the inhomogeneity. It is evident from eqns (7)–(9) that the problem of determining the electroelastic fields in an ellipsoidal inclusion or inhomogeneity is reduced to the problem of determining the piezoelectric Eshelby tensor for an ellipsoidal inclusion.

An inhomogeneous inclusion is an inhomogeneity with prescribed eigenfields Z^T_{Pq}. Consider the infinite piezoelectric solid D with electroelastic moduli E_{iJMn} which contains an ellipsoidal inhomogeneity Ω with electroelastic moduli E^*_{ijkl} and eigenfields Z^T_{Pq}. The stress and electric displacement in the inhomogeneous inclusion are:

$$\Sigma^0_{iJ}+\Sigma_{iJ} = E^*_{iJMn}[Z_{Mn}-Z^T_{Mn}] = E_{iJMn}[Z_{Mn}-Z^T_{Mn}-Z^{**}_{Mn}] = E_{iJMn}[Z_{Mn}-Z^*_{Mn}]. \tag{10}$$

In eqn (10) $Z^*_{Mn} = Z^T_{Mn}+Z^{**}_{Mn}$ where Z^{**}_{Mn} are fictitious eigenfields and $Z_{Mn} = S_{MnAb}Z^*_{Ab}$.

The above results for the interior electroelastic fields can be used to obtain the electroelastic fields just outside an inclusion (and thus of course for an inhomogeneity) by making use of the continuity conditions on Z_{Mn} and the jump conditions on U_M at the inclusion-matrix interface. The fields just outside the inclusion can be expressed as (Dunn and Taya, 1994):

$$\Sigma_{iJ}^{out} = \Sigma_{iJ}^{in} + E_{iJKl}[-E_{pQMn}Z_{Mn}^{*}K_{QK}^{-1}n_p n_l - Z_{Kl}^{*}]. \tag{11}$$

In eqn (11) the interior fields Σ_{iJ}^{in} are obtained by the approach discussed above and K_{QK}^{-1} is the inverse of $K_{JK} = K_{KJ} = n_i n_l E_{iJKl}$ where n_i is the outward normal from the inclusion surface.

To conclude this section we discuss some energy calculations. Consider a piezoelectric solid containing an inhomogeneity subjected to far-field electroelastic loads $\Sigma_{iJ}^{0}\, n_i$. These loads would result in a uniform fields Σ_{iJ}^{0} in a homogeneous solid. The total free energy of the inhomogeneous piezoelectric solid can be expressed as:

$$W = \frac{1}{2}\int_D \Sigma_{iJ}^{0} U_{J,i}^{0}\,\mathrm{d}V + \frac{1}{2}\int_\Omega \Sigma_{iJ}^{0} Z_{Ji}^{*}\,\mathrm{d}V - \int_S \Sigma_{iJ}^{0} n_i U_J^{0}\,\mathrm{d}S \tag{12}$$

where V and S denote the volume and surface, respectively, of the piezoelectric solid and Ω denotes the volume of the inhomogeneity. The first two terms represent the sum of the elastic and electric energy, while the last term is the potential energy due to the loading mechanism. The interaction energy between $\Sigma_{iJ}^{0}\, n_i$ and the inhomogeneity is then:

$$\Delta W = W - W^0 = \frac{1}{2}\int_\Omega \Sigma_{iJ}^{0} Z_{Ji}^{*}\,\mathrm{d}V - \int_S \Sigma_{iJ}^{0} n_i U_J\,\mathrm{d}S = -\frac{1}{2}\Sigma_{iJ}^{0} Z_{Ji}^{*} V_\Omega \tag{13}$$

where the volume of the ellipsoid is $V_\Omega = \pi a_1 a_2 a_3$. Other energy expressions can be readily calculated from these results.

3. INFINITE-BODY GREEN'S FUNCTIONS

In linear piezoelectric solids, the electric and elastic response is anisotropic and coupled. Formally, four Green's functions $G_{IJ}(\mathbf{x}-\mathbf{x}')$ exist which describe the elastic displacement and electric potential at $\mathbf{x}$ due to a point force f_i and point charge Q at $\mathbf{x}'$ (Deeg, 1980; Dunn and Taya, 1993): defining $F_J = (f_1, f_2, f_3, -Q)$, we have $U_M = G_{MJ}F_J$.

We recently derived explicit, closed-form expressions for the infinite-body Green's functions for a transversely isotropic piezoelectric solid (Dunn and Wienecke, 1996). Using the boundedness conditions of that paper we recast the Green's functions into an equivalent form more suitable for the integrations that follow (reverting to x, y, z notation):

$$G_{11} = D_0 \frac{x^2 R_0^2 - y^2 z_0^2}{(x^2+y^2)^2 R_0} - \sum_{i=1}^{3} D_i \lambda_i^{uv} \frac{y^2 R_i^2 - x^2 z_i^2}{(x^2+y^2)^2 R_i}$$

$$G_{12} = G_{21} = D_0 \frac{xy[x^2+y^2+2z_0^2]}{(x^2+y^2)^2 R_0} + \sum_{i=1}^{3} D_i \lambda_i^{uv} \frac{xy[x^2+y^2+2z_i^2]}{(x^2+y^2)^2 R_i}$$

$$G_{13} = G_{31} = \sum_{i=1}^{3} -B_i \lambda_i^{uv} \frac{xz_i}{(x^2+y^2)R_i} = \sum_{i=1}^{3} -D_i \lambda_i^{w} \frac{xz_i}{(x^2+y^2)R_i}$$

$$G_{14} = G_{41} = \sum_{i=1}^{3} A_i \lambda_i^{uv} \frac{xz_i}{(x^2+y^2)R_i} = \sum_{i=1}^{3} -D_i \lambda_i^{\phi} \frac{xz_i}{(x^2+y^2)R_i}$$

$$G_{22} = D_0 \frac{y^2 R_0^2 - x^2 z_0^2}{(x^2+y^2)^2 R_0} - \sum_{i=1}^{3} D_i \lambda_i^{uv} \frac{x^2 R_i^2 - y^2 z_i^2}{(x^2+y^2)^2 R_i}$$

$$G_{23} = G_{32} = \sum_{i=1}^{3} -B_i \lambda_i^{uv} \frac{yz_i}{(x^2+y^2)R_i} = \sum_{i=1}^{3} -D_i \lambda_i^{w} \frac{yz_i}{(x^2+y^2)R_i}$$

$$G_{24} = G_{42} = \sum_{i=1}^{3} A_i \lambda_i^{uv} \frac{yz_i}{(x^2+y^2)R_i} = \sum_{i=1}^{3} -D_i \lambda_i^{\phi} \frac{yz_i}{(x^2+y^2)R_i}$$

$$G_{33} = \sum_{i=1}^{3} B_i \lambda_i^{w} \frac{1}{R_i}$$

$$G_{34} = G_{43} = \sum_{i=1}^{3} -A_i \lambda_i^{w} \frac{1}{R_i} = \sum_{i=1}^{3} B_i \lambda_i^{\phi} \frac{1}{R_i}$$

$$G_{44} = \sum_{i=1}^{3} -A_i \lambda_i^{\phi} \frac{1}{R_i}. \tag{14}$$

In these equations we have set the source point at the origin. The position-dependent terms in eqn (14) are:

$$R_i = \sqrt{x^2+y^2+z_i^2}$$
$$z_i = v_i z. \tag{15}$$

The rest of the terms are functions only of the ten material constants describing a transversely isotropic piezoelectric material and are given as follows:

$$A_1 = \frac{1}{4\pi\gamma_e} \frac{(v_1^2-1)(v_2^2-1)(v_3^2-1)}{v_1(v_1^2-v_2^2)(v_1^2-v_3^2)}$$

$$B_1 = \frac{1}{2\pi\gamma_a}(v_1^2-1)[n_2^e \lambda_3^{uv}(v_3^2-1) - n_3^e \lambda_2^{uv}(v_2^2-1)]$$

$$D_0 = \frac{1}{4\pi C_{44} v_0}$$

$$D_1 = \frac{1}{4\pi\gamma_t} \frac{(\lambda_2^{\phi}\lambda_3^{w} - \lambda_3^{\phi}\lambda_2^{w})}{C_{44}}. \tag{16}$$

The constants A_2 (B_2, D_2) and A_3 (B_3, D_3) are obtained from A_1 (B_1, D_1) by cyclically permuting the indices 1, 2 and 3 and:

$$\gamma_a = (v_1^2-1)\lambda_1^{uv}(n_2^a n_3^e - n_3^a n_2^e) + (v_2^2-1)\lambda_2^{uv}(n_3^a n_1^e - n_1^a n_3^e) + (v_3^2-1)\lambda_3^{uv}(n_1^a n_2^e - n_2^a n_1^e)$$

$$\gamma_e = (\kappa_{11}-\kappa_{33})[C_{11}(C_{44}-C_{33}) + C_{44}(C_{33}+2C_{13}) + C_{13}^2] + C_{11}(e_{33}-e_{15})^2$$
$$+ C_{33}(e_{31}+e_{15})^2 - C_{44}(e_{33}+e_{31})^2 + 2C_{13}[e_{15}(e_{15}+e_{31}-e_{33}) - e_{33}e_{31}]$$

$$\gamma_t = v_1\lambda_1^{uv}(\lambda_3^{\phi}\lambda_2^{w} - \lambda_2^{\phi}\lambda_3^{w}) + v_2\lambda_2^{uv}(\lambda_1^{\phi}\lambda_3^{w} - \lambda_3^{\phi}\lambda_1^{w}) + v_3\lambda_3^{uv}(\lambda_2^{\phi}\lambda_1^{w} - \lambda_1^{\phi}\lambda_2^{w}) \tag{17}$$

$$n_i^a = 2[\lambda_i^{uv}(C_{13}+C_{44}v_i^2) + v_i\lambda_i^{w}(C_{44}-C_{33}) + v_i\lambda_i^{\phi}(e_{15}\text{-}e_{33})]$$
$$n_i^e = 2[-\lambda_i^{uv}(e_{15}v_i^2+e_{31}) + v_i\lambda_i^{w}(e_{33}-e_{15}) + v_i\lambda_i^{\phi}(\kappa_{11}\text{-}\kappa_{33})] \tag{18}$$

$$\lambda_i^{uv} = [(C_{13}+C_{44})e_{33} - C_{33}(e_{31}+e_{15})]v_i^3 + (C_{44}e_{31} - C_{13}e_{15})v_i$$
$$\lambda_i^{w} = -C_{44}e_{33}v_i^4 - [e_{31}(C_{13}+C_{44}) - e_{33}C_{11} + e_{15}C_{13}]v_i^2 - e_{15}C_{11}$$
$$\lambda_i^{\phi} = C_{44}C_{33}v_i^4 + [C_{13}(C_{13}+2C_{44}) - C_{11}C_{33}]v_i^2 + C_{44}C_{11} \tag{19}$$

$v_0 = \sqrt{C_{66}/C_{44}}$ and $-1/v_1^2$, $-1/v_2^2$, and $-1/v_3^2$ are the roots of the cubic equation:

$$s^3+\frac{a}{d}s^2+\frac{b}{d}s+\frac{c}{d}=0 \tag{20}$$

where:

$$\begin{aligned}
a &= C_{11}(\kappa_{11}C_{33}+2e_{15}e_{33})-\kappa_{11}C_{13}(C_{13}+2C_{44})+C_{44}(\kappa_{33}C_{11}+e_{31}^2)-2e_{15}C_{13}(e_{31}+e_{15})\\
b &= C_{33}[\kappa_{11}C_{44}+\kappa_{33}C_{11}+e_{31}(e_{31}+e_{15})]-C_{13}\kappa_{33}(C_{13}+2C_{44})\\
&\quad +(e_{31}+e_{15})(C_{33}e_{15}-2C_{13}e_{33})+e_{33}(C_{11}e_{33}-2C_{44}e_{31})\\
c &= C_{44}(\kappa_{33}C_{33}+e_{33}^2)\\
d &= C_{11}(\kappa_{11}C_{44}+e_{15}^2).
\end{aligned} \tag{21}$$

4. ESHELBY TENSORS

This section contains the principal results of our work: the derivation of the Eshelby tensors for spheroidal inclusions in transversely isotropic piezoelectric solids. To obtain the closed-form expressions for S_{MnAb} we start with the following expression for the displacement and electric potential in a transformed inclusion (Dunn and Taya, 1993):

$$\begin{aligned}
U_M(\mathbf{x}) &= \iint_{|\Omega|} G_{MJ}(\mathbf{x}-\mathbf{x}')\Sigma_{iJ}^* n_i\,\mathrm{d}S(\mathbf{x}')-\iiint_{\Omega} G_{MJ}(\mathbf{x}-\mathbf{x}')\Sigma_{iJ,i}^*\,\mathrm{d}V(\mathbf{x}')\\
&= -E_{iJAb}Z_{Ab}^*\iiint_{\Omega} G_{MJ,i}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}V(\mathbf{x}')
\end{aligned} \tag{22}$$

where the differentiation is with respect to $\mathbf{x}$. In the following we will differentiate U_M with respect to $\mathbf{x}$ to obtain the strain and electric field, focusing on points $\mathbf{x}$ in the inclusion. We will simplify and evaluate the volume integral in a manner analogous to that used by Eshelby (1957) and Withers (1989) for elastic inclusions.

For $\mathbf{x}$ in a convex inclusion we can express the differential volume element as $\mathrm{d}V(\mathbf{x}')=\mathrm{d}r\,\mathrm{d}S=r^2\,\mathrm{d}r\,\mathrm{d}\omega$ in terms of the surface element $\mathrm{d}S$ and the solid angle $\mathrm{d}\omega$ where $r=|\mathbf{x}'-\mathbf{x}|$. It is useful to express $E_{iJAb}G_{MJ,i}(\mathbf{x}-\mathbf{x}')$ in terms of a unit vector $\mathbf{l}=(\mathbf{x}'-\mathbf{x})/|\mathbf{x}'-\mathbf{x}|$:

$$E_{iJAb}G_{MJ,i}(\mathbf{x}-\mathbf{x}') = -\frac{E_{iJAb}g_{MJi}(\mathbf{l})}{r^2} \tag{23}$$

where $g_{MJi}(\mathbf{l})$ is simply the restriction of $G_{MJ,i}(\mathbf{x}-\mathbf{x}')$ to the unit sphere and where the change of sign arises because $G_{MJ,i}(\mathbf{x}-\mathbf{x}')$ is an odd function. Substituting eqn (23) into eqn (22) yields

$$U_M(\mathbf{x}) = Z_{Ab}^*\iiint_{\Omega} E_{iJAb}g_{MJi}(\mathbf{l})\,\mathrm{d}r\,\mathrm{d}\omega. \tag{24}$$

Integrating with respect to r yields

$$U_M(\mathbf{x}) = Z_{Ab}^*\int E_{iJAb}g_{MJi}(\mathbf{l})r(\mathbf{l})\,\mathrm{d}\omega \tag{25}$$

where $r(\mathbf{l})$ defines the boundary of the convex inclusion. For an ellipsoidal inclusion $r(\mathbf{l})$ is given by the positive root of

$$\frac{(x_1+rl_1)^2}{a_1^2}+\frac{(x_2+rl_2)^2}{a_2^2}+\frac{(x_3+rl_3)^2}{a_3^2}=1 \tag{26}$$

where the cartesian coordinates $\mathbf{x}'$ are chosen so that the ellipsoid is centered at the origin and aligned with the coordinate axis. Thus

$$r(\mathbf{l}) = -\frac{f}{g}+\left(\frac{f^2}{g^2}+\frac{e}{g}\right)^{1/2} \tag{27}$$

with

$$e = 1-\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\right) \quad f=\frac{l_1x_1}{a_1^2}+\frac{l_2x_2}{a_2^2}+\frac{l_3x_3}{a_3^2} \quad g=\frac{l_1^2}{a_1^2}+\frac{l_2^2}{a_2^2}+\frac{l_3^2}{a_3^2}. \tag{28}$$

In eqns (26)–(28) a_i are the principal half-axes of the ellipsoid along the x_i' direction. Since $g_{MJi}(\mathbf{l})$ is odd in $\mathbf{l}$ and the quantity $(f^2/g^2+e/g)^{1/2}$ is even in $\mathbf{l}$, their product will integrate to zero. Taking this into account and substituting eqn (27) into eqn (25), U_M can be expressed as

$$U_M(\mathbf{x}) = x_s Z^*_{Ab}\int\frac{\lambda_s E_{iJAb}g_{MJi}(\mathbf{l})}{g}\,\mathrm{d}\omega \tag{29}$$

where

$$\lambda = \left(\frac{-l_1}{a_1^2},\frac{-l_2}{a_2^2},\frac{-l_3}{a_3^2}\right). \tag{30}$$

We can now differentiate eqn (29) to obtain the displacement and potential gradients:

$$U_{M,r} = Z^*_{Ab}\int\frac{\lambda_r E_{iJAb}g_{MJi}(\mathbf{l})}{g}\,\mathrm{d}\omega. \tag{31}$$

Due to linearity we express the strain and potential gradient in terms of a set of piezoelectric Eshelby tensors as:

$$Z_{Mn} = S_{MnAb}Z^*_{Ab}. \tag{32}$$

The set of four tensors S_{MnAb} are thus defined by

$$S_{MnAb} = \begin{cases} \dfrac{1}{2}\displaystyle\int\frac{\lambda_n E_{iJAb}g_{mJi}(\mathbf{l})+\lambda_m E_{iJAb}g_{nJi}(\mathbf{l})}{g}\,\mathrm{d}\omega & M=1,2,3 \\ \displaystyle\int\frac{\lambda_n E_{iJAb}g_{4Ji}(\mathbf{l})}{g}\,\mathrm{d}\omega & M=4 \end{cases}. \tag{33}$$

We remind that the integrals in eqn (33) are over the unit sphere. The task at hand now is to evaluate these integrals in closed-form.

We evaluate the integrals in eqn (33) under the assumption of a spheroidal inclusion ($a_1 = a_2$) where the a_3 axis is normal to the plane of isotropy. This case is quite important as the spheroidal inclusion can model a wide range of microstructural geometry including thin disks, spheres, and long needles. We are now faced with evaluating integrals of the form

$$J_{MniJ} = \int \frac{\lambda_n g_{MJi}(\mathbf{l})}{g} \mathrm{d}\omega. \tag{34}$$

These integrals are closely related to Eshelby's $I(i)$ integrals for elastic inclusions. To evaluate the integrals we write the unit vector $\mathbf{l}$ as $\mathbf{l} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ and g and λ as:

$$g = \frac{a_3^2\sin^2\phi + a_1^2\cos^2\phi}{a_1^2 a_3^2} \qquad -\lambda = \left(\frac{\sin\phi\cos\theta}{a_1^2}, \frac{\sin\phi\sin\theta}{a_1^2}, \frac{\cos\phi}{a_3^2}\right). \tag{35}$$

Upon making these substitutions, eqn (34) can be expressed as:

$$J_{MniJ} = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \frac{-l_n g_{MJi}(\mathbf{l}) a_1^2 a_3^2 \sin\phi}{a_n^2(a_3^2\sin^2\phi + a_1^2\cos^2\phi)} \mathrm{d}\theta\,\mathrm{d}\phi. \tag{36}$$

The integral over θ is easily evaluated and that over ϕ is evaluated after making the substitutions $\tan\phi = v_i \tan\beta$ as done by Withers (1989). All of the integrals that arise in the evaluation of eqn (33) are of the form of eqn (36) and can be evaluated with the same substitutions.

After evaluating these integrals and performing the tensor sums prescribed in eqn (33), we obtain explicit expressions for the Eshelby tensors. Specifically, the non-zero components are:

$$S_{1111} = S_{2222}, S_{1122} = S_{2211}, S_{1133} = S_{2233}, S_{1143} = S_{2243}$$

$$S_{3311} = S_{3322}, S_{3333}, S_{3343}$$

$$S_{4113} = S_{4131} = S_{4223} = S_{4232}, S_{4141} = S_{4242}, S_{4311} = S_{4322}, S_{4333}, S_{4343}$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121}$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232},$$

$$S_{1341} = S_{3141} = S_{2342} = S_{3242}.$$

Explicit expressions for these are:

$$S_{1111} = -C_{66}D_0 J_1(0) + \sum_{i=1}^{3}\left[2\lambda_i^{uv}\left(-C_{13}v_i B_i + (C_{11} - \frac{1}{2}C_{66})D_i\right) - 2e_{31}\lambda_i^{\phi} v_i D_i\right] J_1(i)$$

$$S_{1122} = C_{66}D_0 J_1(0) + \sum_{i=1}^{3}\left[2\lambda_i^{uv}\left(-C_{13}v_i B_i + (C_{11} - \frac{3}{2}C_{66})D_i\right) - 2e_{31}\lambda_i^{\phi} v_i D_i\right] J_1(i)$$

$$S_{1133} = \sum_{i=1}^{3}[2\lambda_i^{uv}(-C_{33}v_i B_i + C_{13}D_i) - 2e_{33}\lambda_i^{\phi} v_i D_i] J_1(i)$$

$$S_{1143} = \sum_{i=1}^{3}[2\lambda_i^{uv}(-e_{33}v_i B_i + e_{31}D_i) + 2\kappa_{33}\lambda_i^{\phi} v_i D_i] J_1(i)$$

$$S_{1212} = -C_{66}D_0 J_1(0) + C_{66}\sum_{i=1}^{3}\lambda_i^{uv} D_i J_1(i)$$

$$S_{1313} = C_{44}v_0 D_0 J_2(0) - \sum_{i=1}^{3} B_i[C_{44}(v_i\lambda_i^{uv} + \lambda_i^{w}) + e_{15}\lambda_i^{\phi}] J_1(i)$$

$$- \sum_{i=1}^{3} [C_{44}\lambda_i^{uv}(v_iD_i+B_i)+e_{15}\lambda_i^{\phi}D_i]J_2(i)$$

$$S_{1341} = e_{15}v_0D_0J_2(0)-\sum_{i=1}^{3} B_i[e_{15}(v_i\lambda_i^{uv}+\lambda_i^{w})-\kappa_{11}\lambda_i^{\phi}]J_1(i)$$

$$- \sum_{i=1}^{3} [e_{15}\lambda_i^{uv}(v_iD_i+B_i)-\kappa_{11}\lambda_i^{\phi}D_i]J_2(i)$$

$$S_{3311} = 4\sum_{i=1}^{3} B_i[(C_{66}-C_{11})\lambda_i^{uv}+e_{31}v_i\lambda_i^{\phi}+C_{13}v_i\lambda_i^{w}]J_2(i)$$

$$S_{3333} = 4\sum_{i=1}^{3} B_i[-C_{13}\lambda_i^{uv}+e_{33}v_i\lambda_i^{\phi}+C_{33}v_i\lambda_i^{w}]J_2(i)$$

$$S_{3343} = 4\sum_{i=1}^{3} B_i[-e_{31}\lambda_i^{uv}-\kappa_{33}v_i\lambda_i^{\phi}+e_{33}v_i\lambda_i^{w}]J_2(i)$$

$$S_{4113} = 2\sum_{i=1}^{3} \lambda_i^{\phi}[e_{15}A_i-C_{44}(B_i+v_iD_i)]J_1(i)$$

$$S_{4141} = -2\sum_{i=1}^{3} \lambda_i^{\phi}[\kappa_{11}A_i+e_{15}(B_i+v_iD_i)]J_1(i)$$

$$S_{4311} = 4\sum_{i=1}^{3} \lambda_i^{\phi}[-e_{31}v_iA_i+C_{13}v_iB_i+(C_{66}-C_{11})D_i)]J_2(i)$$

$$S_{4333} = 4\sum_{i=1}^{3} \lambda_i^{\phi}[-e_{33}v_iA_i+C_{33}v_iB_i-C_{13}D_i]J_2(i)$$

$$S_{4343} = 4\sum_{i=1}^{3} \lambda_i^{\phi}[\kappa_{33}v_iA_i+e_{33}v_iB_i-e_{31}D_i]J_2(i) \tag{37}$$

where:

$$J_1(i) = \frac{\pi\alpha}{(v_i^2\alpha^2-1)^{3/2}}\left[\tanh^{-1}\left(\frac{\sqrt{v_i^2\alpha^2-1}}{v_i\alpha}\right)-v_i\alpha\sqrt{v_i^2\alpha^2-1}\right]$$

$$i = (0 \rightarrow 3, \text{no sum}). \tag{38}$$

$$J_2(i) = \frac{\pi}{(v_i^2\alpha^2-1)^{3/2}}\left[v_i\alpha\tanh^{-1}\left(\frac{\sqrt{v_i^2\alpha^2-1}}{v_i\alpha}\right)-\sqrt{v_i^2\alpha^2-1}\right]$$

In eqn (38) we have defined the aspect ratio $\alpha = a_3/a_1$. $J_1(i)$ and $J_2(i)$ are valid for both oblate and prolate spheroids and in general are complex. S_{MnAb}, however, are always real. $J_1(i)$ and $J_2(i)$ correspond to Eshelby's and Withers' $I_1(i)$ and $I_2(i)$ for elastic inclusions. In fact $J_1(i) = -I_1(i)/2$ and $J_2(i) = v_iI_2(i)/4$. The $I_1(i)$ and $I_2(i)$ for elastic inclusions are usually written in two forms: one for prolate spheroids and one for oblate spheroids (and can be so written here), but this is not really necessary. Simplified forms of $J_1(i)$ and $J_2(i)$ for disk-like, spherical, and needle-like inclusions easily follow as limiting cases of eqn (38).

We have verified the correctness of the S_{MnAb} given by eqn (37) by exhaustive comparison to results obtained by numerically evaluating the surface integral expressions for S_{MnAb} of Dunn and Taya (1993). We also showed that in the absence of piezoelectric coupling ($e_{ij} = 0$), the S_{MnAb} reduce to the results of Withers (1989), plus analogous expressions for transversely isotropic electrostatics. This limiting procedure requires considerable manipulation and thus we only discuss it briefly. When $e_{ij} = 0$, one of the v_i, say v_1, reduces to $\sqrt{\kappa_{11}/\kappa_{33}}$ and v_2 and v_3 reduce to the well-known uncoupled transversely isotropic elastic values. The first term in the three-term sum of the S_{MnAb} then vanishes identically for M, $A = 1, 2, 3$, (i.e., the elastic components), and the remaining two terms reduce to the two terms of Withers' solution. Furthermore, from this point, one can use Withers' analysis to show that they then reduce to Eshelby's (1957) results in the uncoupled isotropic limit.

To conclude, we comment on the numerical implementation of our solution, and in particular the potentially problematic (but easily overcome) numerical aspects. For any given values of the ten material constants we can always evaluate v_j, λ_j^k, and n_j^k. We can also always evaluate the $J_j(i)$ because their limits as $v_j\alpha \to 1$ exist. The only problems we face in numerically evaluating either the Green's functions or Eshelby tensors are those combinations of material constants that cause A_j, B_j, or D_j to become infinite. Examination of the cubic eqn (20) shows that $\mathrm{Re}[v_j] > 0$ which precludes $v_j = 0$ or $v_j = -v_k$. So the A_j become infinite only if $\gamma_e = 0$ (which is equivalent to $v_j = 1$ for some $j > 0$) or if $v_j = v_k$ for some $j \neq k$. Substitution of n_j^k and then λ_j^k into B_j shows that B_j becomes infinite when A_j does and also when $\lambda_j^{uv} = \lambda_j^w = \lambda_j^\phi = 0$ for some j. The D_j have no additional degeneracies. Thus there are three degeneracies in the solution: (i) $v_j = 1$ for some $j > 0$, (ii) $v_j = v_k$ for some $j \neq k$, and (iii) $\lambda_j^{uv} = \lambda_j^w = \lambda_j^\phi = 0$ for some j. The first occurs when the material is uncoupled and either mechanically or electrically isotropic. The second occurs when the material is uncoupled and mechanically isotropic. The third occurs when the material is uncoupled. All three can also occur under more general circumstances. It is important to note that these degeneracies present no real obstacle to the practical application of the solution. Even for a degenerate set of the ten material constants, a valid numerical solution can be obtained by slightly perturbing one of the constants to remove the degeneracy. Furthermore, we remind that our analytical examination of the uncoupled limit reveals that our solution tends correctly to the uncoupled solution.

5. CONCLUSION

The principal results of this paper are the closed-form expressions for the four Eshelby tensors for spheroidal inclusions in transversely isotropic piezoelectric solids. These were obtained using our recent expressions for the infinite-body Green's functions in transversely isotropic piezoelectricity. The piezoelectric Eshelby tensors can be used (in the same manner as Eshelby's tensor for elastic inclusions) to solve a wide range of problems in the mechanics and physics of heterogeneous piezoelectric media.

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